Linear Algebra and Vector Calculus

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Contents

1 Matrices

1.1 Rank of a matrix

1.1.1 Rank of a matrix from echelon form

Definition 1.1.1 *The rank of a matrix is the order of largest size non zero minor available in the matrix.*

Method 1.1.1 *The rank can be computed by searching for the non-zero minors starting from the smallest size to largest size in increasing order. Or starting from the largest size to smallest size in decreasing order.*

Example 1.1.1 *Find the rank of the matrix by reducing it into echelon form*

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

Example 1.1.2 *Find the rank of the matrix by reducing it into echelon form*

$$\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$$

Problem 1.1.1 *Find the rank of the matrix by reducing it into echelon form*

$$\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

Problem 1.1.2 *Find the rank of the matrix by reducing it into echelon form*

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

Problem 1.1.3 *Find the rank of the matrix by reducing it into echelon form*

$$\begin{bmatrix} 3 & 2 & 5 & 7 & 12 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$$

Problem 1.1.4 *Find the rank of the matrix by reducing it into echelon form*

$$\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

Problem 1.1.5 *Find the rank of the matrix by reducing it into echelon form*

$$\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$$

Problem 1.1.6 *Find the rank of the matrix by reducing it into echelon form*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & 3 & 4 & 1 & 2 \end{bmatrix}$$

1.1.2 Rank of a matrix from normal form

Definition 1.1.2 *If $A_{m\times n}$, then the normal form of A is*

$$\begin{bmatrix} I_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$$

where

$$\mathbf{I}_r = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{r\times r}$$

and

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Method 1.1.2 *By applying elementary operations, we make the diagonal element of each column 1 and all elements below it will be 0s. This can be done as follows*

$$\frac{R_i}{a_{ii}}, R_{i+1} - a_{i+1}R_i, R_{i+2} - a_{i+2}R_i, \cdots$$

for $i = 1, 2, \ldots, m$. *Alternatively,*

$$\frac{C_j}{a_{jj}}, C_{j+1} - a_{j+1}C_j, R_{j+2} - a_{j+2}R_j, \cdots$$

Example 1.1.3 *Find the rank of the matrix by reducing it into normal form by using elementary transformations*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & 3 & 4 & 1 & 2 \end{bmatrix}$$

Example 1.1.4 *Find the rank of the matrix by reducing it into normal form by using elementary transformations*

$$\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

Problem 1.1.7 *Find the rank of the matrix by reducing it into normal form by using elementary transformations*

$$\begin{bmatrix} 2 & 3 & -2 & 5 & 1 \\ 3 & -1 & 2 & 0 & 4 \\ 4 & -5 & 6 & -5 & 7 \end{bmatrix}$$

Problem 1.1.8 *Find the rank of the matrix by reducing it into normal form by using elementary transformations*

$$\begin{bmatrix} 3 & 1 & 4 & 6 \\ 2 & 1 & 2 & 4 \\ 4 & 2 & 5 & 8 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$

Problem 1.1.9 *Find the rank of the matrix by reducing it into normal form by using elementary transformations*

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 5 & 10 & 15 & 20 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

Problem 1.1.10 *Find the rank of the matrix by reducing it into normal form by using elementary transformations*

$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

Problem 1.1.11 *Find the rank of the matrix by reducing it into normal form by using elementary transformations*

$$\begin{bmatrix} 1 & -7 & 3 & -3 \\ 7 & 20 & -2 & 25 \\ 5 & -2 & 4 & 7 \end{bmatrix}$$

Problem 1.1.12 *Reduce the matrix into normal form and hence find the rank*

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$$

1.2 Inverse by Gauss Jordan Method

Definition 1.2.1 *Let A, B and D are comformable matrices such that*

$$D = AB$$

1. *Every row operation R affects the pre-multiple*

$$R(D) = R(A)B$$

   Further, if $R_1, R_2, \ldots$ are the row operations on D, then

$$\ldots R_2R_1(D) = \ldots R_2R_1(A)B$$

2. *If C is a column operation on D, it affects the post-multiple*

$$C(D) = AC(B)$$

   Further, if $C_1, C_2, \ldots$ are the column operations on D, then

$$\ldots C_2C_1(D) = A\ldots C_2C_1(B)$$

Method 1.2.1 *Consider the matrix*

$$[A : I] = \begin{bmatrix} a_1 & b_1 & c_1 & 1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 1 \end{bmatrix}$$

Apply row operations only and reduce the matrix

$$[I : B] = \begin{bmatrix} 1 & 0 & 0 & a_1^* & b_1^* & c_1^* \\ 0 & 1 & 0 & a_2^* & b_2^* & c_2^* \\ 0 & 0 & 1 & a_3^* & b_3^* & c_3^* \end{bmatrix}$$

Then B is the required inverse A^{-1}.

Example 1.2.1 *Find the inverse of the matrix by elementary row operations*

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Method 1.2.2 *Consider the matrix*

$$[\tfrac{A}{I}] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply column operations only and reduce the matrix

$$[\tfrac{I}{B}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1^* & b_1^* & c_1^* \\ a_2^* & b_2^* & c_2^* \\ a_3^* & b_3^* & c_3^* \end{bmatrix}$$

Then B is the required inverse A^{-1}.

Example 1.2.2 *Find the inverse of the matrix by elementary column operations*

$$\begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$$

Problem 1.2.1 *Find the inverse of the matrix by elementary row operations*

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Problem 1.2.2 *Find the inverse of the matrix by elementary column operations*

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & -4 & 5 \end{bmatrix}$$

Problem 1.2.3 *Find the inverse of the matrix by elementary column operations*

$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & 5 & 6 \\ -1 & -2 & -4 \end{bmatrix}$$

Problem 1.2.4 *Find the inverse of the matrix by elementary column operations*

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Problem 1.2.5 *Find the inverse of the matrix by elementary column operations*

$$\begin{bmatrix} 1 & 2 & 8 \\ 4 & 7 & 6 \\ 9 & 5 & 3 \end{bmatrix}$$

Problem 1.2.6 *Find the inverse of the matrix by elementary column operations*

$$\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

1.3 Homogeneous Linear Equations

Definition 1.3.1 *A set of equations of the form*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

is called homogeneous linear equations if $b_i = 0, i = 1, 2, \ldots, m$. Otherwise, they are non homogeneous

Expressing the equations (1) in matrix form

$$A\,X = 0 \tag{2}$$

where

$A = \{a_{ij}\}_{m\times n}$: coefficient matrix

$0^T = \{0, 0, \ldots, 0\}_{1\times m}$: null vector

$X^T = \{x_1, x_2, \ldots, x_n\}_{1\times n}$: unknown vector to be determined

Method 1.3.1 *A solution to homogeneous is invariant under the elementary row operations on the coefficient matrix. If $X^\star$ is a solution to (2), then $B\,X^\star = 0$ if B is obtained from A only through elementary row operations.*

Step-1 *Consider the coefficient matrix A.*

Step-2 *Apply elementary row operations on A and reduce it to upper triangular form.*

Step-3 Find the rank of A

- *If r(A) is equal to number of unknowns, then* $\mathbf{X} = \mathbf{0}$ *is the unique (trivial) solution*
- *If r(A) is less than number of unknowns, then infinitely many solutions*

Step-4 Expand the equations and use backward substitution method to solve the variables ie last equation is solved first, followed by last but one and so on.

Following cases arise

1. Case A $m = n$: In this case the coefficient matrix A is a square.
   (a) if A is non singular, then $|A| \neq 0$ and rank of A is equal to n, hence unique triveal solution $X = 0$.
   (b) if A is singular, then $|A| < 0$ and rank of A is less than n, hence many solutions.
2. Case B $m < n$: In this case rank of A is always less than number of unknowns, hence infinitely many solutions.
3. Case C $m > n$: The system will have either unique solution or infinitely many solutions depending upon whether rank is n or less than n.

Method 1.3.2 Another method to solve 3x3 system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}$$

A non-trevial solution is

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{c_2a_3 - c_3a_2} = \frac{z}{a_2b_3 - b_2a_3}$$

Sometimes we need to rearrange the equations to get a non-trevial solution.

Example 1.3.1 *Solve the following system of equations*

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x + 3y + z &= 0 \\ 4x + 5y + 4z &= 0 \\ x + y - 2z &= 0 \end{aligned}$$

Example 1.3.2 *Show that the system of equations*

$$\begin{aligned} 2x - 2y + z &= \lambda x \\ 2x - 3y + 2z &= \lambda y \\ 4x + 5y + 4z &= 0 \\ -x + 2y &= 0 \end{aligned}$$

can possess a non-trivial solution only if $\lambda = 1$ *and* $\lambda = -3$ *and obtain the general solution in each case.*

Problem 1.3.1 *For what value of* k *does the system*

$$\begin{aligned} x + 2y - 3z &= 0 \\ 3x + ky - z &= 0 \\ x - 2y + z &= 0 \end{aligned}$$

possess a non-trivial solution and find the solution for that k.

Problem 1.3.2 *Find the real values of λ for which the equations*

$$x + 2y + 3z = \lambda x 3x + y + 2z = \lambda y 2x + 3y + z = \lambda z$$

have non zero solution.

Problem 1.3.3 *Solve the following system of equations*

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x + 3y + 4z &= 0 \\ 7x + 13y + 19z &= 0 \end{aligned}$$

Problem 1.3.4 *For what value of λ does the following system of equations*

$$\begin{aligned} 2x+3y+z &= 0\\ x+2y+z &= 0\\ 3x+4y+\lambda z &= 0 \end{aligned}$$

have non-zero solution and hence find the solution.

Problem 1.3.5 *For what value of λ does the following system of equations*

$$\begin{aligned} x+y+3z &= 0\\ 4x+3y+\lambda z &= 0\\ 2x+y+2z &= 0 \end{aligned}$$

have non-zero solution and hence find the solution.

Problem 1.3.6 *For what value(s) of λ does the following system of equations*

$$\begin{aligned} 3x+y-\lambda z &= 0\\ 4x-2y-3z &= 0\\ 2\lambda x+4y+\lambda z &= 0 \end{aligned}$$

have non-zero solution and hence find the solution for $\lambda = 1$.

1.4 Non Homogeneous Linear Equations

Definition 1.4.1 *A set of equations of the form*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{3}$$

is called homogeneous linear equations if atleast one $b_i \neq 0, i = 1, 2, \ldots, m$.

Expressing the equations (1) in matrix form

$$A\,X = b \tag{4}$$

where

$A = \{a_{ij}\}_{m\times n}$: coefficient matrix

$b^T = \{b_1, b_2, \ldots, b_m\}_{1\times m}$: RHS vector

$X^T = \{x_1, x_2, \ldots, x_n\}_{1\times n}$: unknown vector to be determined

Definition 1.4.2 Augment Matrix
The matrix $[A : b]$ *is called augment matrix.*

A linear non homogeneous system is completely specified by the augment matrix

Method 1.4.1 *A solution to non homogeneous is invariant under the elementary row operations on the* **augment matrix**. *If* $X^\star$ *is a solution to (2), then* $B\,X^\star = b^\star$ *if* B *is obtained from* A *only through elementary row operations.*

Step-1 *Consider the augment matrix* $[A : b]$.

Step-2 *Apply elementary row operations and reduce it to upper triangular form.*

Step-3 *Find the rank of* A *and rank of* $[A : b]$.

- *If* $r(A) = r([A : b])$, *then the system has solution and called* **consistent**. *Otherwise the system has no solution i.e.* **inconsistent**.
- *If* $r(A) = r([A : b]) = r$ *is less than number of unknowns (n) , then* **infinitely many solutions**.
- *If* $r(A) = r([A : b]) = n$ *the system will have* **unique solution**

Step-4 *Expand the equations and use backward substitution method to solve the variables ie last equation is solved first, followed by last but one and so on.*

Example 1.4.1 *Appy Gauss method to solve the following equations*

$$\begin{aligned} x - 2y + 3z &= 5 \\ 4x + 3y + 4z &= 7 \\ x + y - z &= -4. \end{aligned}$$

Example 1.4.2 *Solve following equations by Gauss method*

$$\begin{aligned} x + y - z &= 0 \\ 2x - y + z &= 3 \\ 4x + 2y - 2z &= 2 \end{aligned}$$

Problem 1.4.1 *Check whether the following equations are consistent, if so solve*

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 + x_3 &= 0 \\ 5x_1 + 3x_2 + 2x_3 &= 4 \end{aligned}$$

Problem 1.4.2 *Apply Gauss elimination method to solve the following equations*

$$\begin{aligned} x + 2y + 3z &= 2 \\ 2x + 4y + 5z &= 3 \\ 3x + 5y + 6z &= 4 \end{aligned}$$

Problem 1.4.3 *Find λ and μ such that*

$$\begin{aligned} x + 2y + \lambda z &= 1 \\ x + 2\lambda y + z &= \mu \\ \lambda x + 2y + z &= 1 \end{aligned}$$

have (i) no solution (ii) unique solution (iii) many solutions.

Problem 1.4.4 *For what values of a and b the following equations*

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 2z &= 6 \\ x + ay + 3z &= b \end{aligned}$$

have (i) no solution (ii) unique solution (iii) many solutions.

Problem 1.4.5 *Apply Gauss Jordan method to solve the following equations*

$$\begin{aligned} 2x - y + 3z &= 9 \\ x + y + z &= 6 \\ x - y + z &= 2. \end{aligned}$$

Problem 1.4.6 *Apply Gauss Jordan method to solve the following equations*

$$\begin{aligned} x + 2y + 3z &= 2 \\ 2x + 4y + 5z &= 3 \\ 3x + 5y + 6z &= 4. \end{aligned}$$

1.5 Iterative Methods

In these methods we assume an initial solution and solve for the next approximate solution. If the solution is satisfactory we stop otherwise we take the latest solution as next initial solution and continue so forth.

1.5.1 Jacobi Method

Let the system of equations to be solved is

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3. \end{aligned}$$

Using first equation for x, second for y and third for z we get

$$x = \frac{1}{a_1}\left[d_1 - b_1 y - c_1 z\right].$$

Similarly,

$$y = \frac{1}{b_2}\left[d_2 - a_2 x - c_2 z\right]$$

and

$$z = \frac{1}{c_3}\left[d_3 - a_3 x - b_3 y\right].$$

1. Let x_0, y_0, z_0 be initial solution and x_1, y_1, z_2 be the next approximate solution.

2. The iterative solution is

$$\begin{aligned} x_1 &= \tfrac{1}{a_1}\left[d_1 - b_1 y_0 - c_1 z_0\right] \\ y_1 &= \tfrac{1}{b_2}\left[d_2 - a_2 x_0 - c_2 z_0\right] \\ z_1 &= \tfrac{1}{c_3}\left[d_3 - a_3 x_0 - b_3 y_0\right] \end{aligned}$$

3. If maximum difference in x, y and z is less than 10^{-3}, then x_1, y_1, z_2 is the solution STOP.

4. If maximum iterations have been completed, then the method didn't converge then x_1, y_1, z_2 is the solution STOP.

5. $x_0 = x_1$, $y_0 = y_1$ and $z_0 = z_1$ and continue.

Table 1: Iterations by Jacobi Method

$Variables$	0	1	2	3
$x_1 = \frac{1}{a_1}\left[d_1 - b_1 y_0 - c_1 z_0\right]$				
$y_1 = \frac{1}{b_2}\left[d_2 - a_2 x_0 - c_2 z_0\right]$				
$z_1 = \frac{1}{c_3}\left[d_3 - a_3 x_0 - b_3 y_0\right]$				

Note: The above iterations converge only when the system of equations are diagonally dominant. i.e $|a_1| > |b_1| + |c_1|$, $|b_2| > |a_2| + |c_2|$ and $|c_3| > |a_3| + |b_3|$. If not the equations have to be rearranged as an initial step.

Example 1.5.1 *Solve the following*

$$\begin{aligned} 8x - 3y + 2z &= 20 \\ 4x + 11y - z &= 33 \\ 6x + 3y + 12z &= 35 \end{aligned}$$

by Jacobi method starting from $x_0 = 0, y_0 = 0, z_0 = 0$ *correct upto three decimal places.*

The given equations are diagonally dominant because $8 > |-3| + |2|$,$11 > |4| + |-1|$ and $12 > |6| + |3|$. Solving for x,y and z respectively we get

$$\begin{aligned} x_1 &= \frac{1}{8}(20 + 3y_0 - 2z_0) \\ y_1 &= \frac{1}{11}(33 - 4x_0 + z_0) \\ z_1 &= \frac{1}{12}(35 - 6x_0 - 3y_0) \end{aligned}$$

Using the initial values $x_0 = 0, y_0 = 0, z_0 = 0$

$$\begin{aligned} x_1 &= \frac{20}{8} = 2.500 \\ y_1 &= \frac{33}{11} = 3.000 \\ z_1 &= \frac{35}{12} = 2.917 \end{aligned}$$

Further iterations are continued in the following table

Table 2: Iterations of Jacobi Method

Vars	0	1	2	3	4	5	6	7	8	9	10
x1	0	2.500	2.896	3.154	3.041	3.017	3.010	3.016	3.017	3.017	3.017
y1	0	3.000	2.356	2.030	1.933	1.970	1.986	1.989	1.986	1.986	1.986
z1	0	2.917	0.917	0.880	0.832	0.913	0.916	0.915	0.911	0.912	0.912

Since the values at 9^{th} and 10^{th} coincide first three decimal places, the approximate solution is $x = 3.017, y = 1.986$ and $z = 0.912$.

Example 1.5.2 *Solve the previous example problem with a new starting solution* $x_0 = 1, y_0 = 1, z_0 = 1$

Problem 1.5.1 *Solve the following problem by Gauss Jacobi method correct upto two decimal places starting from* $x_0 = 1, y_0 = 1, z_0 = 1$

$$\begin{aligned} 8x + y + z &= 8 \\ 2x + 4y + z &= 4 \\ x + 3y + 5z &= 5 \end{aligned}$$

Table 3: Iterations of Gauss Siedal Method

Vars	0	1	2	3	4	5	6	7	8	9	10
x0	1										
y0	1										
z0	1										

Table 4: Iterations of Jacobi Method

Vars	0	1	2	3	4	5	6	7	8	9	10
x0	0										
y0	0										
z0	0										

Problem 1.5.2 *Solve the following problem by Gauss Jacobi method correct upto two decimal places starting from* $x_0 = 1, y_0 = 1, z_0 = 1$

$$\begin{aligned} 8x + y + z &= 8 \\ 2x + 4y + z &= 4 \\ x + 3y + 5z &= 5 \end{aligned}$$

Table 5: Iterations of Jacobi Method

Vars	0	1	2	3	4	5	6	7	8	9	10
x0	0	1	0.75	0.944	0.841	0.898	0.865	0.883	0.872	0.878	0.875
y0	0	1	0.25	0.575	0.353	0.463	0.396	0.432	0.411	0.423	0.416
z0	0	1	0.20	0.700	0.466	0.620	0.543	0.589	0.564	0.579	0.571

Problem 1.5.3 *Solve the following problem by Jacobi method correct upto two decimal places starting from* $x_0 = 0, y_0 = 0, z_0 = 0$

$$\begin{aligned} 5x + 2y + z &= 12 \\ x + 4y + 2z &= 15 \\ x + 2y + z &= 29 \end{aligned}$$

Table 6: Iterations of Jacobi Method

Vars	0	1	2	3	4	5	6	7	8	9	10
x0	0										
y0	0										
z0	0										

1.5.2 Gauss Seidal Method

This method is improved over Gauss Jacobi method in the sense that here we use the latest available values of the variables are used instead of the values obtained in the previous iteration. The iterated equations are suitably modified as follows

$$\begin{aligned} x_1 &= \tfrac{1}{a_1}\left[d_1 - b_1 y_0 - c_1 z_0\right] \\ y_1 &= \tfrac{1}{b_2}\left[d_2 - a_2 x_1 - c_2 z_0\right] \\ z_1 &= \tfrac{1}{c_3}\left[d_3 - a_3 x_1 - b_3 y_1\right] \end{aligned}$$

The initialization and stopping rules are same as those used in Jacobi method.

Example 1.5.3 *Solve the following problem by Gauss Siedal method correct upto two decimal places starting from* $x_0 = 0, y_0 = 0, z_0 = 0$

$$\begin{aligned} 8x + y + z &= 8 \\ 2x + 4y + z &= 4 \\ x + 3y + 5z &= 5 \end{aligned}$$

Table 7: Iterations of Gauss Seidal Method

Vars	0	1	2	3	4	5
x0	0	1.0	0.875	0.875	0.876	0.876
y0	0	0.5	0.438	0.422	0.419	0.419
z0	0	0.5	0.562	0.572	0.573	0.574

Problem 1.5.4 *Solve the following problem by Gauss Siedal method correct upto two decimal places starting from* $x_0 = 0, y_0 = 0, z_0 = 0$

$$\begin{aligned} 5x + 2y + z &= 12 \\ x + 4y + 2z &= 15 \\ x + 2y + z &= 20 \end{aligned}$$

[Ans: x=1, y=2, z=3]

Table 8: Iterations of Gauss Siedal Method

Vars	0	1	2	3	4	5	6	7	8	9	10
x0	0										
y0	0										
z0	0										